university of groningen / faculty of science and engineering

WMIE003-05, WBIE043-05	Block IIa, 2025-26
Product Design by Finite Element Method	27-02-2026
Individual midterm examination	Time: 60 minutes

Student Name: ____

S-Number: ____

Midterm exam blueprint (60 min, 50 pts)

Question	Format	Points	Suggested time
Q1	MCQ (7×1)	07	15 min
Q2	Worked problem	20	20 min
Q3	Worked problem	13	15 min
Q4	Worked problem	10	10 min
	Total	**50**	**60 min**

Q1 Multiple-choice questions (07 points)

Choose exactly **one** option (A–D) per item. No penalty for wrong answers.

1. The “primary” unknown solved at nodes in basic structural FEM is typically:

A. Stress

B. Strain

C. DisplacementD. Young’s Modulus

2. A global stiffness matrix becomes singular primarily when:

A. The mesh is too fine
B. Essential boundary conditions are insufficient (rigid mode present)
C. There are too many loads
D. Young’s Modulus is too high

3. For Newtonian fluid, shear stress is related to shear rate by:

A. $Density\ i.e. \tau = \rho\dot{\gamma}$
B. $Viscosity\ i.e. \tau = \mu\dot{\gamma}$
C. $Young'sModulus\ i.e. \tau = E\dot{\gamma}$
D. $Posion's\ ratio\ i.e. \tau = \dot{\gamma}/v$

4. In idealized Couette flow (parallel plates, no pressure gradient), the velocity profile is:

A. Uniform

B. Linear in the gap direction

C. Parabolic

D. Exponential

5. A bar with both ends fully fixed loaded at an interior node typically yields:

A. zero reactions

B. reactions at both ends

C. no stress

D. singular matrix always

6. For a simply supported beam AB with a single downward point load W, the vertical reactions satisfy:

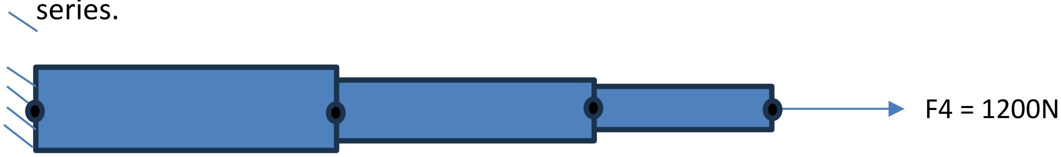
A. $R_A - R_B = W$
B. $R_A + R_B = W$
C. $R_A = R_B$ always
D. $R_A + R_B = 0$

7. Drag force magnitude is commonly modeled as:

A. $F_D = \rho U c_d A$
B. $F_D = \frac{1}{2}\rho U^2 c_d A$
C. $F_D = \mu U c_d A$
D. $F_D = \frac{1}{2}\mu U^2 c_d A$

Q2 Worked problem: 1D bar FEM with three elements (20 points)

A stepped bar (with one end fully fixed) is modeled with **three** axial bar (truss) elements in series.




```
Node 1          Node 2          Node 3          Node 4
u1 = 0          u2 =?             u3 =?           u4 =?
```

`Fixed support at Node 1, No external load at node 2 and 2 (F2 = F3 = 0)`

Given (use N–mm–MPa):

Element 1: $L_1 = 300$ mm, $A_1 = 80$ mm^2, $E_1 = 70{,}000$ N/mm^2

Element 2: $L_2 = 200$ mm, $A_2 = 40$ mm^2, $E_2 = 210{,}000$ N/mm^2

Element 3: $L_3 = 100$ mm, $A_3 = 20$ mm^2, $E_3 = 150{,}000$ N/mm^2

Nodal loads: F_1 unknown (reaction), $F_2 = F_3 = 0$, $F_4 = +1200$ N - Boundary condition: $u_1 = 0$

Tasks:

a) Compute each element stiffness $k_e = \frac{EA}{L}$. (4 pts)

b) Assemble the global stiffness matrix K (4×4). (5 pts)

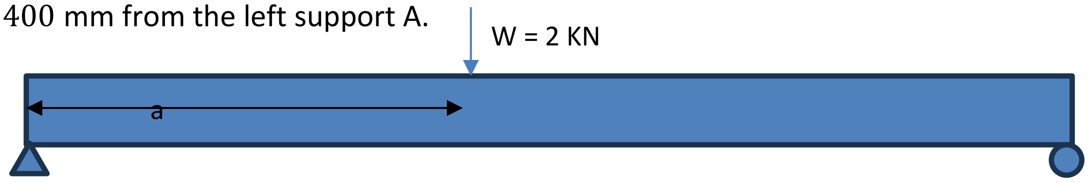
c) Apply BCs and solve for u_2, u_3 and u_4. (5 pts)

d) Compute the reaction force at Node 1. (2 pts)

e) Compute axial stress in each element and state tension/compression. (4 pts)

Q3 Worked problem: simply supported beam reactions and bending stress (13 points)

A simply supported beam of span $L = 1200$ mm carries a point load $W = 2000$ N at $a = 400$ mm from the left support A.



`A (pin) ---- a = 400 mm ---- ↓ W = 2000 N ---- (L-a) =800 mm ---- B (roller)`

Cross-section: rectangle with width $b = 30$ mm and height $h = 60$ mm.

Tasks:

a) Compute the reactions R_A and R_B. (4 pts)
b) Determine the maximum bending moment M_{max} and where it occurs. (4 pts)
c) Compute the maximum bending stress magnitude $\sigma_{\text{max}} = \frac{M_{\text{max}} y}{I}$ with $I = \frac{bh^3}{12}$, $y = \frac{h}{2}$. State whether the **top** fiber is in tension or compression. (5 pts)

Q4 Worked problem: plane Couette flow between plates (10 points)

A Newtonian fluid fills the gap between two large parallel plates separated by $h = 1.0$ mm. The top plate moves at constant speed $U = 0.50$ m/s; the bottom plate is stationary. Assume steady laminar Couette flow with no pressure gradient.

Given: dynamic viscosity $\mu = 0.10$ Pa·s; density $\rho = 1000$ kg/m^3; plate area $A = 0.020$ m^2.

Tasks:

a) Compute the shear rate $\dot{\gamma} = U/h$. (2 pts)

b) Compute shear stress $\tau = \mu\dot{\gamma}$. (2 pts)

c) Compute the force F required to pull the top plate: $F = \tau A$. (3 pts)

d) Compute Reynolds number using $Re = \rho U h/\mu$ and comment “laminar plausible: yes/no.” (3 pts)

Solution

Q1 Multiple-choice questions (07 Points)

Choose exactly **one** option (A–D) per item. No penalty for wrong answers.

Q1.1 Primary unknown solved at nodes in basic structural FEM
Correct: **C. Displacement**.
Reasoning: In displacement-based structural FEM, nodal DOFs are typically displacements; strains/stresses are derived from displacement gradients and constitutive law. This is standard in displacement-based FEM formulations.

Q1.2 A global stiffness matrix becomes singular primarily when
Correct: **B. Essential boundary conditions are insufficient (rigid mode present)**.
Reasoning: If rigid-body motion is not prevented by essential BCs, there exists a nonzero displacement vector producing zero strain energy → K is rank-deficient (singular).

Q1.3 For Newtonian fluid, shear stress related to shear rate by
Correct: **B.** $\tau = \mu\dot{\gamma}$.
Reasoning: Newtonian constitutive law states shear stress is proportional to shear rate, constant of proportionality is dynamic viscosity μ.

Q1.4 In idealized Couette flow (parallel plates, no pressure gradient), velocity profile
Correct: **B. Linear in the gap direction**.
Reasoning: Couette flow between plates with one moving wall and no pressure gradient yields a linear velocity profile.

Q1.5 A bar with both ends fully fixed loaded at an interior node typically yields
Correct: **B. reactions at both ends**.
Reasoning: Fixed–fixed supports create two reaction forces; an interior applied load is shared between supports depending on stiffness distribution. This also aligns with the stiffness-method examples in early structural FEM.

Q1.6 Simply supported beam with a single downward point load W: reactions satisfy
Correct: **B.** $R_A + R_B = W$.
Reasoning: Vertical force equilibrium of the whole beam gives $R_A + R_B - W = 0$.

Q1.7 Drag force magnitude is commonly modeled as
Correct: **B.** $F_D = 1/2\, \rho U^2 c_d A$.
Reasoning: Standard drag model uses dynamic pressure $1/2\, \rho U^2$ multiplied by reference area and drag coefficient.

Q2 Worked problem: 1D bar FEM with two elements (20 points)

Problem summary (from exam): Stepped axial bar with **3 elements** in series (nodes 1–4). Node 1 fixed $u_1 = 0$. Loads: $F_2 = F_3 = 0$, $F_4 = +1200$ N. Compute element stiffnesses, assemble global K (4×4), solve for u_2, u_3, u_4, reaction F_1, and stresses in each element.

Given (N–mm–MPa system)
- Element 1: $L_1 = 300$ mm, $A_1 = 80$ mm², $E_1 = 70{,}000$ N/mm²
- Element 2: $L_2 = 200$ mm, $A_2 = 40$ mm², $E_2 = 210{,}000$ N/mm²
- Element 3: $L_3 = 100$ mm, $A_3 = 20$ mm², $E_3 = 150{,}000$ N/mm²

Element stiffness for a 2-node axial bar in stiffness form is standard: $k = \frac{EA}{L}$,

$K_e = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$.

Step 1: Element stiffnesses (4 pts)

$$k_e = \frac{E_e A_e}{L_e}$$

- Element 1:

  $k_1 = \frac{70{,}000 \cdot 80}{300} = 18{,}666.67$ N/mm
- Element 2:

  $k_2 = \frac{210{,}000 \cdot 40}{200} = 42{,}000$ N/mm
- Element 3:

  $k_3 = \frac{150{,}000 \cdot 20}{100} = 30{,}000$ N/mm

Rubric (4 pts)
- 1.5 pts: correct k_1 with units
- 1.5 pts: correct k_2 with units
- 1.0 pt: correct k_3 with units

Follow-through: if one stiffness is numerically wrong but formula and units are correct, award most method credit (e.g., 70–80% of that element's points).

Step 2: Assemble global stiffness matrix K (5 pts)

Node ordering: $[u_1, u_2, u_3, u_4]^T$. For a serial 1D bar chain:

$$K = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix}$$

Substitute $k_1 = 18666.67,\ k_2 = 42000,\ k_3 = 30000$:

$$K = \begin{bmatrix} 18666.67 & -18666.67 & 0 & 0 \\ -18666.67 & 60666.67 & -42000 & 0 \\ 0 & -42000 & 72000 & -30000 \\ 0 & 0 & -30000 & 30000 \end{bmatrix} \text{N/mm}$$

Rubric (5 pts)
- 2 pts: correct tridiagonal structure and signs
- 2 pts: correct diagonal assemblies $(k_1 + k_2)$, $(k_2 + k_3)$
- 1 pt: correct numeric substitution

Step 3: Apply BCs and solve for u_2, u_3, u_4 (5 pts)

Load vector (applied nodal loads):

$$F = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \text{unknown} \\ 0 \\ 0 \\ 1200 \end{bmatrix}$$

Apply essential BC $u_1 = 0$. Reduced unknown vector: $\{u_r\} = [u_2, u_3, u_4]^T$. Reduced system:

$$\underbrace{\begin{bmatrix} 60666.67 & -42000 & 0 \\ -42000 & 72000 & -30000 \\ 0 & -30000 & 30000 \end{bmatrix}}_{K_r} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1200 \end{bmatrix}$$

Solve by back-substitution:

From row 3:

$$-30000u_3 + 30000u_4 = 1200 \Rightarrow u_4 - u_3 = 0.04 \text{ mm} \Rightarrow u_4 = u_3 + 0.04$$

Row 2:

$$-42000u_2 + 72000u_3 - 30000u_4 = 0$$

Substitute $u_4 = u_3 + 0.04$:

$$-42000u_2 + 72000u_3 - 30000(u_3 + 0.04) = 0$$

$$-42000u_2 + 42000u_3 - 1200 = 0 \Rightarrow u_3 - u_2 = 0.0285714 \Rightarrow u_3 = u_2 + 0.0285714$$

Row 1:

$$60666.67u_2 - 42000u_3 = 0 \Rightarrow 60666.67u_2 = 42000u_3 \Rightarrow u_2 = 0.6923077u_3$$

Combine with $u_3 = u_2 + 0.0285714$. Substitute $u_2 = 0.6923u_3$:

$$u_3 = 0.6923u_3 + 0.0285714 \Rightarrow 0.3077u_3 = 0.0285714 \Rightarrow u_3 = 0.0928571 \text{ mm}$$

Then

$$u_2 = 0.0642857 \text{ mm}, \quad u_4 = u_3 + 0.04 = 0.1328571 \text{ mm}$$

Rubric (5 pts)
- 2 pts: correct reduced system $K_r u_r = F_r$
- 3 pts: correct u_2, u_3, u_4 (1 pt each; ±2% tolerance)
Follow-through: if a stiffness was wrong, award the 2 setup points plus proportional follow-through on the solved displacements if algebra is consistent.

Step 4: Reaction at node 1 (2 pts)

Element 1 connects nodes 1–2, so its local nodal force vector is:

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}_{(e1)} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

So, the nodal force at node 1 contributed by element 1 is:

$$F_1 = k_1(u_1 - u_2)$$

Substitute $u_1 = 0$, $u_2 = 0.0642857$ mm, $k_1 = 18666.67$ N/mm:

$$F_1 = 18666.67(0 - 0.0642857) = -1200 \text{ N}$$

Reaction at node 1 (external):

$$F_1 = -1200 \text{ N}$$

Interpretation: the support force acts in the $-x$ direction to balance the applied $+1200$ N at node 4.

Rubric (2 pts)

- Correct back-substitution expression at node 1: $F_1 = k_1(u_1 - u_2)$: 1 pt
- Correct numeric value and sign $F_1 = -1200$ N: 1 pt

Step 5: Element stress and tension/compression (4 pts)

Because there are **no external nodal forces at nodes 2 and 3** ($F_2 = F_3 = 0$), the internal axial force is constant through the series bar at equilibrium (a useful reasoning check). We can compute element force directly from FE displacements:

$$N_e = k_e(u_j - u_i)$$

- Element 1 (1–2):
  $N_1 = k_1(u_2 - u_1) = 18666.67(0.0642857) = 1200$ N
- Element 2 (2–3):
  $N_2 = k_2(u_3 - u_2) = 42000(0.0285714) = 1200$ N
- Element 3 (3–4):
  $N_3 = k_3(u_4 - u_3) = 30000(0.04) = 1200$ N

Stress $\sigma = N/A$ (MPa because 1 N/mm^2 $= 1$ MPa):

- $\sigma_1 = 1200/80 = 15$ MPa (tension)
- $\sigma_2 = 1200/40 = 30$ MPa (tension)
- $\sigma_3 = 1200/20 = 60$ MPa (tension)

Concise interpretation: **element 3 has the highest stress** because it has the smallest cross-sectional area, even though the axial force is the same in all series elements.

Rubric (4 pts)
- 2 pts: correct element forces (or correct statement "same 1200 N through each element" + justification)
- 2 pts: correct stresses + T/C labels (≈0.67 pt each element, including sign/tension)

Q3 Worked problem solution: simply supported beam reactions and bending stress (13 Points)

Given
Single point load $W = 2000$ N at $a = 400$ mm from left support A. Diagram gives $(L - a) =$ 800 mm → $L = 1200$ mm. Cross-section: $b = 30$ mm, $h = 60$ mm.

Bending stress formula $\sigma = My/I$, rectangle $I = bh^3/12$, was introduced in structural lecture.

Step 1: Reactions R_A, R_B (4 pts)

Sum of moments about A:

$$R_B L - Wa = 0 \Rightarrow R_B = \frac{Wa}{L} = \frac{2000 \cdot 400}{1200} = 666.67 \text{ N}$$

Vertical force equilibrium:

$$R_A + R_B = W \Rightarrow R_A = 2000 - 666.67 = 1333.33 \text{ N}$$

Rubric (4 pts)
- 2 pts correct R_B (equations + value)
- 2 pts correct R_A (equations + value)

Step 2: Maximum bending moment M_{max} and location (4 pts)

For a simply supported beam with a single point load, the shear changes sign at the load; the maximum moment occurs **at the load location** $x = a$. Compute:

$$M_{\max} = R_A a = 1333.33 \cdot 400 = 533{,}333 \text{ N·mm}$$

Location: $x = 400$ mm from A.

Rubric (4 pts)
- 2 pts correct location reasoning
- 2 pts correct $M_{\max}$

Step 3: Maximum bending stress and top-fiber sign (5 pts)

Second moment of area:

$$I = \frac{bh^3}{12} = \frac{30 \cdot 60^3}{12} = \frac{30 \cdot 216{,}000}{12} = 540{,}000 \text{ mm}^4$$

Extreme-fiber distance:

$$y = \frac{h}{2} = 30 \text{ mm}$$

Bending stress magnitude:

$$\sigma_{\max} = \frac{M_{\max} y}{I} = \frac{533{,}333 \cdot 30}{540{,}000} = 29.63 \text{ MPa}$$

For sagging moment under a downward load, the top fiber is in **compression** (bottom in tension).

Rubric (5 pts)
- 2 pts: correct I and y
- 2 pts: correct $\sigma_{\max}$ (±2%)
- 1 pt: correct "top in compression" statement (with brief justification)

Q4 Worked problem solution: plane Couette flow between plates (10 Points)

Given (from exam)
Plate gap $h = 1.0$ mm, top plate speed $U = 0.50$ m/s, $\mu = 0.10$ Pa·s, $\rho = 1000$ kg/m^3, area $A = 0.020$ m^2.

Couette flow relations (linear profile, $\dot{\gamma} = U/h$, $\tau = \mu\dot{\gamma}$) are canonical for Newtonian flow between plates. [7]

Convert: $h = 1.0$ mm $= 1.0 \times 10^{-3}$ m.

Step 1: Shear rate $\dot{\gamma} = U/h$ (2 pts)

$$\dot{\gamma} = \frac{U}{h} = \frac{0.50}{1.0 \times 10^{-3}} = 500 \text{ s}^{-1}$$

Rubric (2 pts)
- 1 pt correct formula and unit conversion
- 1 pt correct numeric value

Step 2: Shear stress $\tau = \mu\dot{\gamma}$ (2 pts)

$$\tau = \mu\dot{\gamma} = 0.10 \times 500 = 50 \text{ Pa}$$

Rubric (2 pts)
- 1 pt correct formula
- 1 pt correct value with units

Step 3: Force required $F = \tau A$ (3 pts)

$$F = \tau A = 50 \times 0.020 = 1.0 \text{ N}$$

Rubric (3 pts)
- 1 pt correct expression
- 2 pts correct computation + units

Step 4: Reynolds number and laminar plausibility (3 pts)

Using the exam's definition $Re = \rho U h/\mu$:

$$Re = \frac{\rho U h}{\mu} = \frac{1000 \cdot 0.50 \cdot 1.0 \times 10^{-3}}{0.10} = 5$$

Interpretation: $Re \approx 5$ is very small → laminar is **plausible** (yes).

Rubric (3 pts)
- 2 pts correct Re computation (including unit conversion)
- 1 pt correct plausibility statement ("laminar: yes") with brief reasoning